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Note

A Variable Coefficient Extension of a Formula for Series Conversion

1. Introduction

A five-term recurrence formula for converting a series of polynomials $\sum_{m=0}^{n} a_m q_m(x)$ into $\sum_{m=0}^{n} A_m Q_m(x)$ when $q_m \equiv q_m(x)$ and $Q_m \equiv Q_m(x)$ satisfy three-term recurrence formulas

$$q_{-1} = 0, \quad q_{m+1} + (a(m) + b(m)x)\, q_m + c(m)\, q_{m-1} = 0, \quad m = 0(1)\, n-1, \qquad \text{(1a)}$$

and

$$Q_{-1} = 0, \quad Q_{m+1} + (A(m) + B(m)x)\, Q_m + C(m)\, Q_{m-1} = 0, \quad m = 0(1)\, n-1, \qquad \text{(1b)}$$

has been given for the case where the coefficients a_m and A_m are constant (derivation in [1]; more compact expression in [2, 3]). The derivation was based upon a well-known recurrence scheme of Clenshaw for summing the series $\sum_{m=0}^{n} a_m q_m(x)$ when $q_m(x)$ is any kind of function, not necessarily polynomial, which satisfies a recurrence formula $q_{m+1} + \alpha(m, x)\, q_m + \beta(m, x)\, q_{m-1} = 0$ with no restrictions on the form of the functions $\alpha(m, x)$ and $\beta(m, x)$ [4]. It was first noted in [5] that Clenshaw makes no statement in [4] to imply that in his summation scheme the coefficients a_m need not be constant with respect to x (also noted subsequently in [1], and indicated independently by Ng in [6] who wrote $a_m(x)$ for a_m). This present note gives an extension of that five-term recurrence formula for converting

$$\sum_{m=0}^{n} a_m(x)\, q_m(x) \qquad \text{into} \qquad \sum_{m=0}^{n_1} A_m Q_m(x), \qquad (2)$$

for q_m and Q_m polynomials satisfying (1a) and (1b), $a_m(x)$ being a polynomial but not restricted to the mth degree, and A_m still constant, so that $n_1 \geqslant n$.

2. Extended Formula

We suppose that each $a_m(x)$, of degree $d_m \geqslant 0$, has been expressed as

$$a_m(x) = \sum_{i=0}^{d_m} a_{m,i} Q_i(x). \qquad (3)$$

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Then A_m, $m = 0(1)\, n_1$, is equal to $a_m^{(n)}$ obtained from the recurrence formula

$$\begin{aligned} a_m^{(k+1)} = & -a_m^{(k-1)} c(n-k) + a_{m-1}^{(k)} b(n-k-1)/B(m-1) \\ & + a_m^{(k)}[-a(n-k-1) + b(n-k-1)\, A(m)/B(m)] \\ & + a_{m+1}^{(k)} b(n-k-1)\, C(m+1)/B(m+1) + a_{n-k-1,m}, \end{aligned} \tag{4}$$

where

$$m = 0(1)\, N_{k+1}, \qquad k = -1(1) \qquad n-1, \tag{4a}$$

$$\begin{gathered} N_{-1} = 0, \qquad N_0 = d_n, \qquad N_{k+1} = \max[N_k + 1, N_{k-1}, d_{n-k-1}], \\ k \geqslant 0, \qquad \text{going up to} \quad N_n = n_1, \end{gathered} \tag{4b}$$

$$a_i^{(j)} = 0 \qquad \text{for} \quad i < 0, \qquad \text{or} \qquad i > N_j, \tag{4c}$$

and

$$a_{n-k-1,m} = 0, \qquad m > d_{n-k-1}. \tag{4d}$$

The derivation of (4) is entirely similar to that given in [1] for constant a_m's, but we now take into account the degree N_i in the $(n-i)$th term in Clenshaw summation by backward recurrence [4].

3. Computational Advantage

Of course, if we first obtain for every term in $\sum_{m=0}^{n} a_m(x)\, q_m(x)$ the expression for the right member of

$$a_m(x)\, q_m(x) = \sum_{j=m_1}^{m_2} b_{m,j} q_j(x), \tag{5}$$

$b_{m,j}$ constant, $m_1 \leqslant m \leqslant m_2$, through the repeated use of (1a), we may then apply the recurrence formula given in [1–3] for constant coefficients, for $n = n_1$. Discounting the difference in the preliminary work of obtaining the right members of (3) and (5), the conversion with constant coefficients involves the calculation of an $n_1 + 1$ by $n_1 + 1$ triangle of $a_m^{(k)}$'s, in number $\sim n_1^2/2$, or, if $n_1 \sim 2n$, $\sim 2n^2$, while (4) with variable coefficients $a_m(x)$ involves $\sum_{j=0}^{n} (N_j + 1)$ quantities $a_m^{(k)}$, whose number will generally be much less than $(n+1) \times$ largest $(N_j + 1) = (n+1)(n_1+1)$, say $\sim 3n^2/2$ by assuming $N_j \sim 3n/2$ on the average, which is likely to be even less in many cases.

4. Further Extension

We may still apply (4) to the case where the coefficients $a_m(x)$ are not polynomials, but given approximately by (3) for sufficiently high degrees d_m and hope to obtain a sufficiently close approximation to the coefficients A'_m, $m = 0(1)n_1$, in the infinite series right member of

$$\sum_{m=0}^{n} a_m(x)\, q_m(x) = \sum_{m=0}^{\infty} A'_m Q_m(x), \tag{6}$$

from the coefficients A_m in the series $\sum_{m=0}^{n_1} A_m Q_m(x)$.

5. Conversion of Products

A useful special application of (4) occurs in the direct conversion of the product of two series, namely,

$$\sum_{i=0}^{N} b_i P_i(x) \times \sum_{m=0}^{n} a_m q_m(x) \qquad \text{into} \qquad \sum_{m=0}^{n+N} A_m Q_m(x), \tag{7}$$

for b_i, a_m, A_m constants, and $P_i(x)$ any ith degree polynomials not required to satisfy recurrence formulas. Taking $a_m(x)$ in (2) as $a_m \sum_{i=0}^{N} b_i P_i(x)$, they are identical except for a constant factor so that when $\sum_{i=0}^{N} b_i P_i(x)$ is expressed as $\sum_{i=0}^{N} B_i Q_i(x)$, we have $d_m = N$, $a_{m,i} = a_m B_i$, for $m = 0(1)n$, in (3). This application includes products $\phi(x) \sum_{m=0}^{n} a_m q_m(x)$ for $\phi(x)$ any function that is closely approximable by an Nth degree polynomial.

As one instance of a practical example involving (7), we might wish to obtain a Q_m series for $\phi(x) f(x)$ when $f(x)$ is approximated by some interpolation series $\sum_{m=0}^{n} a_m q_m(x)$. In particular, when the Q_m's are Chebyshev polynomials adjusted to any range $[a, b]$, we may replace the conversion formula given in [3, (1c)] by the present (4) applied to the interpolation polynomials considered in [3]. It may be true that, in the case where the Q_m's are Chebyshev polynomials, the advantage in applying (4) to (7) does not appear too great over the alternative method of obtaining the right side of (7) from the product of the Chebyshev series for $\phi(x)$ and $f(x)$, replacing each $T_r(x)\, T_s(x)$ by simple and well-known expressions $\sum c_i T_i(x)$. However, for other important Q_m's, e.g., Legendre, Jacobi, Laguerre, or Hermite polynomials, where the needed expressions for $Q_r Q_s$ as $\sum c_i Q_i(x)$ are not so simple or familiar, the use of (4) appears to be more advantageous.

Incidentally, the preceding remarks suggest another very special use for (7), which is to obtain the product of two Q_m series (constant coefficients) as a Q_m series, i.e., converting

$$\sum_{i=0}^{N} B_i Q_i(x) \times \sum_{m=0}^{n} a_m Q_m(x) \qquad \text{into} \qquad \sum_{m=0}^{n+N} A_m Q_m(x), \tag{8}$$

where $d_m = N$ and $a_{m,i} = a_m B_i$, $m = 0(1)n$, as before. But now the $c(n-k)$, $b(n-k-1)$, and $a(n-k-1)$ in (4) are replaced by $C(n-k)$, $B(n-k-1)$ and $A(n-k-1)$, respectively. When the first factor on the left side of (8) is unity, (4) degenerates into its original unextended version [3,(1c)] which is formulated only so that $\sum_{m=0}^{n} a_m Q_m(x)$ reproduces itself.

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